Topics in complex analysis. Lecture 8 (Nov 11, 2024)

Proposition 5.10. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic such that $\{0,1\} \cap f(G) = \emptyset$. Then there exists a holomorphic function $h : G \to \mathbb{C}$ such that* (1)

$$f = \frac{1}{2}(1 + \cos(\pi \cos(\pi h))) .$$

If $\widetilde{h} : G \to \mathbb{C}$ is any holomorphic function satisfying the above equation, then $\widetilde{h}(G)$ contains no disc of radius larger or equal than 1. (2)

__Proof__ (1) For the repr formula: the fct $2f-1$ omits the values ± 1. Hence applying Lemma 5.9 yields $\exists\, h_1 : G \to \mathbb{C}$ holo:

$$2f - 1 = \cos \pi h_1$$

Since $2f-1 \neq \pm 1$ necessarily h_1 omits all integer values, in part ± 1. Apply Lemma 5.9 to h_1 to infer $\exists\, h : G \to \mathbb{C}$ holo:

$$h_1 = \cos \pi h .$$

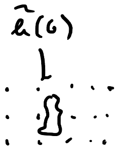
Rearrange terms to obtain the repr. formula

(2) Let $\tilde{h} : G \to \mathbb{C}$ any holo fct st $f = \frac{1}{2}(1 + \cos(\pi \cos(\pi \tilde{h})))$

Define the grid

$$\Omega := \left\{ m \pm i\pi^{-1} \log\left(n + \sqrt{n^2-1}\right) : m \in \mathbb{Z},\ n \in \mathbb{N} \right\}$$

$0 \notin \mathbb{N}$



We claim (a) $\tilde{h}(G) \cap \Omega = \emptyset$

(b) The grid size (horizontal and vertical) doesn't exceed 1.

$\therefore$ no Ball of radius ≥ 1 can fit into $\tilde{h}(G)$.

__Proof of (a)__ A calculation shows

$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$$\cos\left(\pi\left(m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right)\right)\right) = (-1)^m n \in \mathbb{Z}$$

Applying $\cos(\pi\,\cdot)$ to this yields $\cos(\pi \cos(\pi\,\cdot)) = \pm 1$ on Ω

Since f omits the values 0 and 1, necessarily $\tilde{h}(G) \cap \Omega = \emptyset$.

(in other words: $\tilde{h}$ doesn't attain any value on the grid Ω)

keep m fixed, increase n

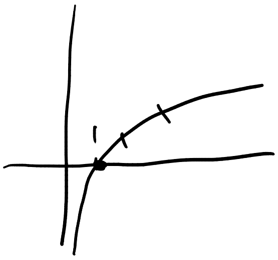
Proof of (b) The "vertical" difference between neighboring grid points is

$$\frac{1}{\pi}\left| \log\left(n+1+\sqrt{(n+1)^2-1}\right) - \log\left(n+\sqrt{n^2-1}\right)\right|$$

$$= \frac{1}{\pi}\left| \log\left(\frac{1+1/n+\sqrt{1+2/n}}{1+\sqrt{1-1/n^2}}\right)\right|$$

use log rules and cancel an n in the fraction

$$= \frac{1}{\pi}\left(\log\left(1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}\right) - \underbrace{\log\left(1+\sqrt{1-\frac{1}{n^2}}\right)}_{\geq 0}\right)$$



$$\leq \frac{1}{\pi}\log\left(1+\underbrace{\frac{1}{n}}_{\leq 1}+\sqrt{1+\underbrace{\frac{2}{n}}_{\leq 2}}\right)$$

$$\leq \frac{1}{\pi}\log\left(2+\sqrt{3}\right) < 1$$

The "horizontal" difference is clearly 1.

keep n fixed, increase m

□

Theorem 5.11 (Schottky's theorem). *Let $f \in \mathcal{H}(\overline{B_1(0)})$ be such that $|f(0)| \leq r$ and $\{0,1\} \cap f(\overline{B_1(0)}) = \emptyset$. Then*

$$|f(z)| \leq L(\theta, r) \qquad \forall |z| \leq \theta,\ 0 < \theta < 1.$$

$$L(\theta, r) := \exp\left(\pi \exp\left(\pi\left(3 + 2r + \frac{\theta}{\beta(1-\theta)}\right)\right)\right)$$

Proof Step 1. (a) We claim $\cos(\pi a) = \cos(\pi b) \implies b = \pm a + 2n$ for some $n \in \mathbb{Z}$

(b) $\forall w \in \mathbb{C}\ \exists v \in \mathbb{C}$ st $\cos(\pi v) = w$ & $|v| \leq 1 + |w|$.

Proof of step 1 (a) Recall

$$\cos(\pi a) - \cos(\pi b) = -2\sin\left(\frac{\pi}{2}(a+b)\right)\sin\left(\frac{\pi}{2}(a-b)\right)$$

The lhs is 0 iff $\sin\left(\frac{\pi}{2}(a+b)\right) = 0$ or $\sin\left(\frac{\pi}{2}(a-b)\right) = 0$

Since sin vanishes precisely at $\pi\mathbb{Z}$ the claim follows

(eg if $\sin\left(\frac{\pi}{2}(a+b)\right) = 0$ this means $a+b \in 2\mathbb{Z}$)

(b) Since cos is surjective $\forall w \in \mathbb{C}$ $\exists v \in \mathbb{C}$ with $\underline{\mathrm{Re}\, v \in [-1,1]}$ st $\cos(\pi \underline{v}) = w$.

$v \longmapsto \cos(\pi v)$ 2-periodic

Recall $|w|^2 = \cos^2(\pi \,\mathrm{Re}\, v) + \sinh^2(\pi\, \mathrm{Im}(v))$.

↳ use the formula $\cos(\pi v) = \cos(\pi \,\mathrm{Re}\, v) \cosh(\pi \,\mathrm{Im}\, v) - i \sin(\pi \,\mathrm{Re}\, v) \sinh(\pi \,\mathrm{Im}\, v)$ & trigonometric identities.

& $\sinh^2 x \geq x^2 \quad \forall x \in \mathbb{R}$

Hence

$$|v| = \sqrt{\underbrace{(\mathrm{Re}\, v)^2}_{\leq 1} + (\mathrm{Im}\, v)^2}$$

sqrt subadd

$$\leq \sqrt{1 + \frac{1}{\pi^2} \underbrace{\pi^2 (\mathrm{Im}\, v)^2}_{\leq \sinh^2(\pi \,\mathrm{Im}\, v) \leq \sinh^2(\pi \,\mathrm{Im}\, v) + \cos^2(\pi \,\mathrm{Re}\, v) = |w|^2}} \leq \sqrt{1 + \frac{|w|^2}{\pi^2}} \leq 1 + |w|.$$

<u>Step 2</u> There exists a function $g \in \mathcal{H}(\overline{B}_1(0))$ st

(a) $f = \frac{1}{2}(1 + \cos(\pi \cos(\pi g)))$ and

$$|g(0)| \leq 3 + 2|f(0)|$$

(b) $$|g(z)| \leq |g(0)| + \frac{\vartheta}{\beta(1-\vartheta)} \quad \text{for } z \in \overline{B}_\vartheta(0)$$

where $\vartheta \in (0,1)$.

A const for which the Bloch thm holds, eg. $\frac{3}{2} - \sqrt{2}$

<u>Proof of (a)</u> By Lemma 5, 9, $\exists \tilde{F} \in \mathcal{H}(\overline{B}_1(0))$ st

$$\boxed{2f - 1 = \cos(\pi \tilde{F})}$$

By Step 1 (b) $\exists b \in \mathbb{C}$:

$\cos(\pi \underline{b}) = 2f(0) - 1$ & $|b| \leq 1 + |2f(0) - 1|$

$\cos(\pi \tilde{F}(0))$

$\leq 2 + 2|f(0)|$

Again by Step 1 (a) this implies $b = \pm \tilde{F}(0) + 2k, \quad k \in \mathbb{Z}$.

Define $F := \pm \tilde{F} + 2k \in \mathcal{H}(\bar{B}_1(0))$. Note that

$$2f - 1 \underset{(*)}{=} \cos(\pi F) \quad \& \quad \boxed{F(0) = b.}$$

With the same argument: F necessarily omits all integer values by $(*)$, hence ± 1, hence $\boxed{F = \cos(\pi \tilde{g})}$ for some $\tilde{g} \in \mathcal{H}(\bar{B}_1(0))$ by Lemma 5.9.

By Step 1 (b) $\exists\, a \in \mathbb{C}$: $\cos(\pi a) = b$ &

$$|a| \leq 1 + |b| \leq 1 + 2 + 2|f(0)| = 3 + 2|f(0)|$$

Since $\cos(\pi a) = \cos(\pi \tilde{g}(0))$, we define $g = \pm \tilde{g} + 2m$, where $m \in \mathbb{Z}$ sat. $a = \pm \tilde{g}(0) + 2m$. Then

$$f = \frac{1}{2}\left(1 + \cos(\pi \cos(\pi g))\right) \quad \text{(by constr)}$$

& $\quad |g(0)| \leq 3 + 2|f(0)|$

Proof of (b) By Prop 5.10 $g(B_1(0))$ contains no disk of radius ≥ 1. On the other hand the following holds. (i.e. $g(B_1(0))$ contains disks of some radii)

Homework 7.1 (A general version of Bloch's theorem)**.** Show if $f : U \to \mathbf{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains disks of every radius $(3/2 - \sqrt{2})s|f'(c)|$, where $s \in (0, \operatorname{dist}(c, \partial U))$. In particular, show if $f : \mathbf{C} \to \mathbf{C}$ is entire and nonconstant, then $f(\mathbf{C})$ contains disks of arbitrarily large radii.

$\beta \qquad 1-\vartheta$

$$\forall z \in \bar{B}_\vartheta(0), \quad \operatorname{dist}(z, \partial B_1(0)) \geq 1 - \vartheta$$

In particular: $\quad \beta(1-\vartheta)\,|g'(z)| \leq 1$

Prop 5.10

Hence by fund thm of calculus $\forall\, |z| \le \vartheta$

$$|g(z)| \le |g(z)-g(0)| + |g(0)|$$

$$\le \int_{[0,z]} \underbrace{|g'(\xi)|}_{\le \frac{1}{\beta(1-\vartheta)}} d\xi + |g(0)|$$

$$\le \frac{|z|}{\beta(1-\vartheta)} + |g(0)| \le \frac{\vartheta}{\beta(1-\vartheta)} + |g(0)|.$$

<u>Step 3</u> we use the estimates $|\cos z| \le e^{|z|}$

& $\frac{1}{2}|1+\cos z| \le e^{|z|}$ (triangle inequ)

$$|f(z)| = \left|\frac{1}{2}(1+\cos(\pi\cos(\pi g)))\right|$$

$$\le \exp(\pi \exp(\pi\, |g(z)|))$$

Step 2 $\le \exp\left(\pi\left(\exp\left(\pi\left(\frac{\vartheta}{\beta(1-\vartheta)} + |g(0)|\right)\right)\right)\right)$

Step 1 $\le \exp\left(\pi\left(\exp\left(\pi\left(\frac{\vartheta}{\beta(1-\vartheta)} + 3 + 2|f(0)|\right)\right)\right)\right)$

$\square$

<u>Thm</u> The statement of Picard's great thm is equiv to the following: given any holo $f : B_1(0)\setminus\{0\} \to \mathbb{C}\setminus\{0,1\}$ either f or $1/f$ is bounded around 0.

Theorem 5.12 (Sharpened version of Montel's theorem). *Let $D \subset \mathbb{C}$ be a domain and*

$$\mathcal{F} := \{f : D \to \mathbb{C} \text{ holomorphic}, \{0,1\} \cap f(D) = \emptyset\}.$$

Let $\{f_n\}_{\in \mathbb{N}} \subset \mathcal{F}$. Then either $\{f_n\}$ contains a subsequence that converges locally uniformly to some holomorphic function $f : D \to \mathbb{C}$ or the whole sequence $|f_n|$ converges locally uniformly to $+\infty$.

Proof Step 1 Let $z_0 \in D$ & $r > 0$ and set

$$\mathcal{F}(z_0, r) := \{ f \in \mathcal{F} : |f(z_0)| \le r \}.$$

on a nbhd of z_0

Claim $\mathcal{F}(z_0, r)$ is equibdd. This follows from Schottky's thm.

$$\sup_{f \in \mathcal{F}(z_0,r)} \sup_{z \in B_\delta(z_0)} |f(z)| < \infty$$

chosen appropriately

Step 2 $\mathcal{F}(z_0, r)$ is locally bdd in D.

Let $\mathcal{U} := \{ z \in D : \mathcal{F}(z_0, r) \text{ equibdd on a nbhd of } z \}$.

Claim $\mathcal{U}$ is both open & closed in D, hence all of D.

Openness: by step 1, $z_0 \in \mathcal{U}$. and it's open.

Closedness: by contradiction assume $w \in \partial\mathcal{U} \cap D$ st. $\mathcal{F}(z_0, r)$ not bdd in any nbhd of w.

By Step 1 $\exists\, (f_n) \subseteq \mathcal{F}(z_0, r)$ st $\lim_{n\to\infty} |f_n(w)| = \infty$

Def. $g_n = 1/f_n$ well-defined in a nbhd of w and

$$\lim_{n \to \infty} |g_n(w)| = 0 \qquad (*)$$

This implies $(g_n) \subseteq \mathcal{F}(w, R)$ for suitable $R > 0$ (Cauchy integral formula). Applying Montel's Thm 1.7 a subseque. converges on a smaller disk $B_r(w)$ loc unif to $g : B_r(w) \longrightarrow \mathbb{C}$ $r < R$.

Since g_n has no zero $\forall n$ and $(*)$, we infer $g \equiv 0$ (Corollary 1.6).

since $B_r(w) \cap \mathcal{U} \neq \emptyset$ this means $\overline{\lim_n} |f_n(z)| = +\infty$ for some $z \in \mathcal{U}$. ↯ to the def of $\mathcal{U}$.

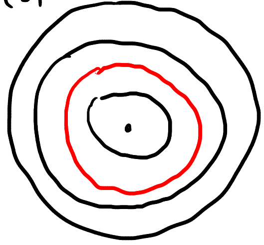
<u>Step 3</u> Apply Step 2 and Montel's thm either to f_n if infinitely many f_n belong to $\mathcal{F}(z_0, 1)$ and $1/f_n$ otherwise. $\square$

<u>Proof</u> of Picard's great thm. From exercises, we know the concl of PGT is equiv to: for every holo fct $f : B_1(0) \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0,1\}$, f or $1/f$ is bounded in a nbhd of 0.

Given such f, we define $f_n : B_1(0) \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0,1\}$ by $f_n(z) = f(z/n)$. Apply Thm 5.12 to $(f_n)_{n \in \mathbb{N}}$: for a subsequ, $(f_{n_k})_{k \in \mathbb{N}}$ or $(1/f_{n_k})_{k \in \mathbb{N}}$ is locally unif bdd on $B_1(0) \setminus \{0\}$.

In the first case, $\exists C > 0$: $\sup_{k \in \mathbb{N}} |f(z/n_k)| \leq C$ whenever $|z| = 1/2$

$B_1(0) \setminus \{0\}$



By the maximum principle, $|f(z)| \leq C$ whenever $z \in A_k = \{ z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k} \}$.

Note

$$A := \bigcup_{k \in \mathbb{N}} A_k$$

is a punctured nbhd of 0 in $B_1(0) \setminus \{0\}$ on which f is bdd. Argument in the case of $(1/f_{n_k})_{k \in \mathbb{N}}$ is analogous. $\square$